Many-Body Vortex Dynamics in a Bose-Einstein Condensate

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Introduction

In the recent work [JS], R. L. Jerrard and D. Smets study the **asymptotic dynamics of a vortex state** given by the 2D time-dependent Gross-Pitaevskii (GP) equation in the Thomas-Fermi regime ($\varepsilon \to 0$):

$$\begin{cases} i\varepsilon^{-\frac{4}{s+2}}\partial_t u_t = -\Delta u_t + \frac{1}{\varepsilon^2}\left(V(y) + |u_t|^2\right) u_t \\ u_t(y)|_{t=0} = u(y). \end{cases} \tag{2DTF}$$

The question that naturally arise in this setting is whether one can prove some similar asymptotic at the many-body level **for a 3D gas** confined in a cylindrical trap. In particular we consider a system with short range interactions stronger than in the usual GP scaling. This way we can use the GP asymptotic for the system and **still have some freedom on the order parameter** in front of the nonlinearity to consider the Thomas-Fermi asymptotic of the system and then apply [JS].

The problem

- **Framework and Hypotheses:** We consider the many-body dynamics for a system of trapped bosons with a *compactly supported, spherically symmetric interaction* that scales with the number of particles, with initial datum completely factorized and constant along the cylindrical axis, that is

$$H_N := \sum_{j=1}^{N}\left(-\Delta_j + V(r_j)\right) + \sum_{1\le j<k\le N} R_N N^{3\beta-1} v\left(N^{\beta}(x_j - x_k)\right)$$

$$\begin{cases} i\partial_t \Phi_t = H_N \Phi_t \\ \Phi_t(\mathbf{x})|_{t=0} = \bigotimes_{j=1}^{N} \frac{1}{\sqrt{h}}\,\varphi(r_j). \end{cases} \tag{MBD}$$

- **Our Goal:** We aim at proving that the solution to (MBD) converges in norm to the projector on the solution to (2DTF).

Approximation Steps

Many-Body System

We consider the many-body dynamics given by (MBD) where

- the one-particle Hilbert space is $\mathcal{H} := L^2\left(\mathbb{R}^2 \times [-\frac{h}{2}, \frac{h}{2}]\right)$;
- the Hamiltonian H is defined over functions with *Neumann boundary conditions* on the interval $\left[-\frac{h}{2}, \frac{h}{2}\right]$ (we will anyway send h to infinity so actually the boundary conditions will not matter);
- $v \in L^\infty$, spherically symmetric and compactly supported;
- V depends only on the two dimensional variable, $V(r) = k|r|^s$;
- R_N go to infinity as N goes to infinity and $R_N = o(N)$ (e.g., $R_N = \log(N)$).

From Many-Body to Hartree

Applying a similar analysis as in [P] one can prove that the evolution of the many-body state is close to the solution of the Hartree equation, i.e., if

$$\begin{cases} i\partial\phi_t = -\Delta\phi_t + V(r)\phi_t + R_N\left(N^{3\beta}v(N^{\beta}\cdot) * |\phi_t|^2\right)\phi_t \\ \phi_t(x)|_{t=0} = \frac{1}{\sqrt{h}}\,\varphi(r). \end{cases}$$

then

$$\lim_{N\to+\infty}\left\|\gamma^{\Phi_t} - |\phi_t\rangle\langle\phi_t|\right\|_{\mathrm{op}} = 0$$

uniformly in $0 \le t \le T$ with T depending only on ϕ_0, where γ^{Φ_t} is the one-body reduced density matrix of Φ_t.

From Hartree to Gross-Pitaevskii

The next step is to prove closeness of ϕ_t to the solution of the GP equation:

$$\begin{cases} i\partial_t\psi_t = -\Delta\psi_t + V(r)\psi_t + R_N a\,|\psi_t|^2\,\psi_t \\ \psi_t(x)|_{t=0} = \frac{1}{\sqrt{h}}\,\varphi(r) \end{cases} \tag{3DGP}$$

with $a := \int_{\mathbb{R}^3} dx\ v(x)$. In particular one can prove L^2 convergence of ϕ_t to ψ_t uniformly in $0 \le t \le T$.

Two crucial ingredients:

- the potential $N^{3\beta}v(N^{\beta}x)$ converges as a distribution to a Dirac delta;
- the energy of the solution to (3DGP) is close to the GS energy, i.e.

$$\mathcal{E}[\phi] \le E + \mathcal{O}\left(R_N^{-\frac{2}{s+2}}\log R_N\right) \tag{E}$$

where

$$\mathcal{E}[\phi] := \int_{\mathbb{R}^2} dy\,\left\{|\nabla\phi|^2 + V(r)|\phi|^2 + \frac{R_N a}{2}|\phi|^4\right\}$$

$$E := \inf_{\|\phi\|_2=1}\mathcal{E}[\phi].$$

The latter requirement is actually **necessary** to prove [JS].

From 3D to 2D

Choosing now h going to infinity as N goes to infinity and assuming that

$$\lim_{N\to+\infty} h^{-1}R_N = \infty$$

the three dimensional solution to the GP problem stays close to the following two dimensional problem

$$\begin{cases} i\partial\varphi_t = -\Delta\varphi_t + V(r)\varphi_t + \widetilde{a}_N|\varphi_t|^2\varphi_t \\ \varphi_t(r)|_{t=0} = \varphi(r) \end{cases}$$

where $\widetilde{a}_N := \frac{aR_N}{h}$. Actually $\psi_t(x) = \frac{1}{\sqrt{h}}\,\varphi_t(r)$ thanks to **the uniqueness of the solution** to the GP equation.

Rescaling of the length

To apply the result in [JS] we need the potential term and the nonlinear term to scale in the same way with N, so we rescale the dimensions so that

$$u_t(y) := \xi\varphi_t(r), \quad u(y) := \xi\varphi(r), \quad r = \xi y, \quad \xi := \widetilde{a}_N^{\frac{1}{s+2}}$$

and calling $\varepsilon := \frac{1}{\sqrt{\widetilde{a}_N}}$ we get for u_t to solve (2DTF).

Main Result

Theorem. *Under the assumption* (E) *on the initial datum* ϕ *and defining*

$$u_t^{\mathrm{resc}}(x) := \frac{1}{\sqrt{h}\xi}u_t\left(\frac{r}{\xi}\right)$$

we get

$$\lim_{N\to+\infty}\left\|\gamma^{\Phi_t} - |u_t^{\mathrm{resc}}\rangle\langle u_t^{\mathrm{resc}}|\right\|_{op} = 0$$

where Φ_t *is the solution of* (MBD) *and* u_t *is the solution of* (2DTF).

Remark. *The timescale* T *is actually bigger than the timescale of existence of the vortices as found in [JS].*

Open Questions

In [JS] it is crucial to be able of relate the **measure of the vorticity** of a state to a Dirac delta representing the position of the vortices; this involves a different topology on the GP state. One should wonder how to translate this topology at the level of the many-body system; our current idea is to consider a norm that contains information about the energy of the state too.

References

[JS] R.L. Jerrard, D. Smets, *Vortex dynamics for the two dimensional non homogeneous Gross-Pitaevskii equation*, Annali Scuola Norm. Sup. Pisa, vol. 14, no. 3, 729-766, 2015

[P] P. Pickl, *A Simple Derivation of Mean Field Limits for Quantum Systems* Lett. Math. Phys. 97: 151, 2011